

ON THE RATIONALITY OF THE SPECTRUM

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ABSTRACT. Let $\Omega \subset \mathbb{R}$ be a compact set with measure 1. If there exists a subset $\Lambda \subset \mathbb{R}$ such that the set of exponential functions $E_\Lambda := \{e_\lambda(x) = e^{2\pi i \lambda x}|_\Omega : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\Omega)$, then Λ is called a spectrum for the set Ω . A set Ω is said to tile \mathbb{R} if there exists a set \mathcal{T} such that $\Omega + \mathcal{T} = \mathbb{R}$. A conjecture of Fuglede suggests that Spectra and Tiling sets are related. Lagarias and Wang [14] proved that Tiling sets are always periodic and are rational. That any spectrum is also a periodic set was proved in [3], [8]. In this paper, we give some partial results to support the rationality of the spectrum.

1. INTRODUCTION

In this paper we explore the rationality of the spectrum in \mathbb{R} .

Let $\Omega \subset \mathbb{R}^d$ be a (compact) set with Lebesgue measure $|\Omega| = 1$.

Definition 1. Ω is said to be a spectral set if there exists a subset $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{e_\lambda(x) = e^{2\pi i \lambda \cdot x}|_\Omega : \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space $L^2(\Omega)$. The set Λ is said to be a spectrum for Ω , and the pair (Ω, Λ) is called a spectral pair.

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It is easy to see that for a spectral set Ω , the spectrum need not be unique, and conversely given a spectrum Λ , there may be many sets Ω such that (Ω, Λ) is a spectral pair.

Interest in the spectrum arose from a conjecture due to Fuglede relating spectral and tiling properties of sets. Ω is said to tile \mathbb{R}^d if there exists a subset $\mathcal{T} \subset \mathbb{R}^d$ such that $\Omega + \mathcal{T}$ is a partition (a.e.) of \mathbb{R}^d . (Ω, \mathcal{T}) is called a tiling pair and \mathcal{T} is called a tiling set.

Fuglede's Conjecture[6]. A set $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if Ω tiles \mathbb{R}^d by translations.

The conjecture suggests that there could be a strong relationship between Spectra and Tiling sets for a given Ω .

Fuglede's conjecture is known to be false in dimensions $d \geq 3$ [16] [15] [11]. However interest in this conjecture is still alive. For $d = 2$ it has been proved for convex planar domains for $d = 2$ [9] and for $d = 3$, Fuglede's conjecture for Convex polytopes in \mathbb{R}^3 has been proved very recently by Greenfeld and Lev [7] .

We will restrict to dimension $d = 1$. In this case Fuglede's conjecture is known to hold when Ω is the union of two intervals [12] and for the case of three intervals the authors proved that Tiling implies Spectral, and except for one situation, Spectral implies tiling too, [1], [2] . In Lagarias and Wang [14] studied the structure of tiling sets \mathcal{T} , and proved that if (Ω, \mathcal{T}) is a tiling pair for some Ω , then the tiling set \mathcal{T} is periodic with an integer period and is rational, i.e. \mathcal{T} is of the form

$$\mathcal{T} = \cup_{j=0}^{n-1} (t_j + n\mathbb{Z})$$

with $t_0 = 0$ and $t_j \in \mathbb{Q}$, $j = 0, 1, \dots, n-1$.

Our aim is to study the structure of spectra Λ for spectral pairs (Ω, Λ) . In [3], the authors proved that if Ω is the union of finitely many intervals and (Ω, Λ) is a spectral pair, then Λ is periodic with an integer period. In [8] this result was then proved for any compact set Ω .

Let (Ω, Λ) be a spectral pair. Since any translate of Λ is again a spectrum for Ω , we may assume that Λ is of the form

$$\Lambda = \cup_{j=0}^{d-1} (\lambda_j + d\mathbb{Z}) = \Gamma + d\mathbb{Z},$$

with $\lambda_0 = 0$. Further, by the structure theorem proved in [3] we know that Λ is also a spectrum for a set Ω_1 which is a union of d equal intervals, whose end points lie on the lattice \mathbb{Z}/d ; i.e.,

$$\Omega_1 = \cup_0^{d-1} [a_j/d, a_j + 1/d)$$

$j = 0, 1, \dots, d-1$, with $a_0 = 0$. Thus to resolve the question of rationality of a spectrum, it is enough to assume that the associated spectral set Ω is of the above form (such sets are called *clusters*). After rescaling, we write

$$\Omega = A + [0, 1]$$

with $A \subset \mathbb{Z}_+$, $0 \in A$, $|A| = d$. Then $\Lambda = \cup_{j=1}^{d-1} (\lambda_j/d + \mathbb{Z}) = \Gamma + \mathbb{Z}$, and $E_\Lambda = \{\frac{1}{\sqrt{d}}e_\lambda(x), \lambda \in \Lambda\}$ is a complete orthonormal set.

All known spectra of sets in \mathbb{R} are rational; however it is not known whether this must always be so. In [4] it is shown that if Fuglede's conjecture is true in one dimension, then every spectral set of Lebesgue measure 1 has a rational spectrum.

The following result due to Laba [13] is the only result we are aware of in the literature, which addressed the problem of rationality of spectra for clusters:

Theorem 1 (Laba). *Suppose that $\Omega = A + [0, 1]$, $A \subset \mathbb{Z}_+$, $|A| = d$ is a spectral set. If $\Omega \subset [0, M]$, where $M < \frac{5d}{2}$, then all spectra associated to Ω are rational.*

In section 2, we deduce two interesting facts about a spectrum from known results. First we observe that elements of any spectrum are either rational or transcendental; next, we relate the rationality of the spectrum to integer zeros of exponential polynomials. In Section 3, we show that if for some Ω such that (Ω, Λ) is a spectral pair and Ω contains some patterns, then Λ has to be rational. In section 4, we prove that if the set $\Omega - \Omega$ contains some rigid structures, then the spectrum is rational.

2.

2.1. We first observe that the elements of the spectrum are either rational or transcendental. We explain this below:

We have

$$\widehat{\chi_\Omega}(\xi) = (1 - e^{2\pi i \xi})(1 + e^{2\pi i a_1 \xi} + \dots + e^{2\pi i a_{d-1} \xi}),$$

and for every λ_j , $j = 0, 1, \dots, d-1$, $\widehat{\chi_\Omega}(\lambda_j) = 0$. So each $e^{2\pi i \lambda}$ is an algebraic number, in fact, an algebraic integer. Recall the following famous result:

Theorem 2 (Gelfond-Schneider). *If α and β are algebraic numbers with $\alpha \neq 0, 1$, and if β is not a rational number, then any value of $\alpha^\beta = \exp(\beta \log \alpha)$ is a transcendental number.*

We apply this theorem to $\alpha = e^{\pi i} = -1$, and $\beta = 2\lambda_k$. Since $\alpha^\beta = e^{2\pi i \lambda_k}$ is an algebraic integer, $2\lambda_k$ is either rational or is not an

algebraic number. In other words, elements λ_k of the spectrum are either rational or transcendental numbers.

2.2. Zeros of Exponential Polynomials. Consider the tempered distribution obtained as Dirac masses on points of Λ , i.e the distribution

$$\delta_\Lambda = \sum_{j=0}^{d-1} \sum_{n \in \mathbb{Z}} \delta_{n+\lambda_j/d} = \delta_\Gamma * \delta_{\mathbb{Z}}$$

Then

$$\widehat{\delta_\Gamma}(x) = \sum_{j=0}^{d-1} e^{2\pi i \lambda_j x/d}$$

Recall that with Ω and Λ as above, where $|\Omega| = d$, (Ω, Λ) is a spectral pair iff $\frac{1}{d}|\widehat{\chi_\Omega}|^2 * \delta_\Lambda \equiv d$ iff $|\widehat{\chi_\Omega}|^2 * \delta_\Gamma * \delta_{\mathbb{Z}} \equiv d^2$ iff $(\Omega - \Omega)|_{\mathbb{Z}} \subseteq \mathbb{Z}(\widehat{\delta_\Gamma}) \cup \{0\}$, where $\mathbb{Z}(\widehat{\delta_\Gamma}) = \{k \in \mathbb{Z} : \widehat{\delta_\Gamma}(k) = 0\}$.

We write $z_j = e^{2\pi i \lambda_j/d}$, then

$$\widehat{\delta_\Gamma}(k) = \sum_{j=0}^{d-1} z_j^k, \quad k \in \mathbb{Z}$$

We are thus led to study the integer zeros of exponential polynomials; more specifically the integer zeros of exponential polynomials. An important result in this context is the Skolem-Mahler-Lech theorem, which says that the sets of integer zeros of exponential polynomials are of the form $X \cup F$, where X is a union of finitely many complete arithmetic progressions, and F is a finite set.

In [14] Lagarias and Wang considered the case of the exponential polynomial, $\widehat{\delta_\Gamma}$, and gave more precise description of the set X , which we need to state.

With the above rescaling, we have, $\Gamma = \{\lambda_0 = 0, \lambda_1/d, \dots, \lambda_{d-1}/d\}$. we write $\gamma_j = \lambda_j$, and define an equivalence relation on Γ as $\gamma_i \sim \gamma_j$ iff

$\gamma_i - \gamma_j \in \mathbb{Q}$, and we partition Γ into its rational equivalence classes,

$$\Gamma = \bigcup_1^k \Gamma_j^*$$

Then

$$\widehat{\delta}_\Gamma(\xi) = \widehat{\delta}_{\Gamma_1^*}(\xi) + \cdots + \widehat{\delta}_{\Gamma_k^*}(\xi)$$

where,

$$\Gamma_j^* = \left\{ \gamma_j, \gamma_j + \frac{l_{j,2}}{m_j}, \gamma_j + \frac{l_{k,3}}{m_j}, \dots, \gamma_j + \frac{l_{j,n_j}}{m_j} \right\}$$

Notice

$$\widehat{\delta}_{\Gamma_j^*}(\xi) = e^{2\pi i \gamma_j \xi} (1 + e^{2\pi i \frac{l_{j,2}}{m_j} \xi} + \cdots + e^{2\pi i \frac{l_{j,n_j}}{m_j} \xi})$$

Hence if

$$\widehat{\delta}_{\Gamma_j^*}(\xi_0) = 0 \text{ then } \widehat{\delta}_{\Gamma_j^*}(\xi_0 + m_j p) = 0, \forall p \in \mathbb{Z}$$

Thus the zero set of $\widehat{\delta}_{\Gamma_j^*}$ is m_j periodic.

Let $M := LCM \{m_1, m_2, \dots, m_k\}$. Let X be the common integer zero set of $\{\widehat{\delta}_{\Gamma_j^*}\}$ i.e.

$$X := \bigcap_{i=1}^k \mathbb{Z}(\widehat{\delta}_{\Gamma_j^*})$$

Then X is M periodic and $X \subseteq \mathbb{Z}(\widehat{\delta}_\Gamma)$, and $F := \mathbb{Z}(\widehat{\delta}_\Gamma) \setminus X$ is a finite set.

As a consequence, we prove:

Theorem 3. *Let (Ω, Λ) be a spectral pair as above. Then Λ is rational if and only if $(A - A) \cap F = \emptyset$.*

Proof. Clearly if Λ is rational, then $F = \emptyset$. For the converse, we have

$$\chi_\Omega * \chi_\Omega \cdot \widehat{\delta}_\Gamma \cdot \delta_{\mathbb{Z}} = d^2$$

Consider the equivalence class Γ_1^* , for which

$$\chi_\Omega * \chi_\Omega \cdot \widehat{\delta_{\Gamma_1}} \cdot \delta_{\mathbb{Z}} = d|\Gamma_1| \leq d^2.$$

But

$$|\widehat{\chi_\Omega}|^2 * \delta_{\Gamma_1^*} * \delta_{\mathbb{Z}}(0) = d^2$$

Hence $|\Gamma_1^*| = d$, so that there is only one rational class. \square

Remark 1.

- (1) *For the Skolem-Mahler-Lech theorem, the structure and cardinality of the finite set F have been studied (see [5], and references therein), but "effective" results are largely unknown [5], [17].*
- (2) *In the case when each of the equivalence classes Γ_j^* are singletons, i.e. $\lambda_j - \lambda_k \notin \mathbb{Q}$ for all $j \neq k$, then it can be easily seen that in fact $\mathbb{Z}(\widehat{\delta_\Gamma}) = F$. (See Corollary 1.20 in [5]).*

3.

In this section we show that the existence of some specific structures or patterns (which we call *flags*) in the zero set $\mathbb{Z}(\widehat{\delta_\Gamma})$ guarantees the rationality of Γ .

We begin with a result which follows from a result of Jager [10]. We state it in our setting and give a proof for the sake completeness. (Jager's paper is difficult to find).

Definition 2. *For fixed integers m , r and $N > r$, let $S_0 = \{m+1, m+2, \dots, m+r\}$ and for an $N \geq r$ let $S_n = S_{n-1} + N$, $n = 1, 2, \dots, s-1$. These S_n 's are called *strips* and the set $F = \cup_0^{s-1} S_n$ is called an $r \times s$ -flag. We will think of a flag as an array:*

$$\begin{array}{cccc}
m+1 & m+2 & \cdots & m+r \\
\circ & \circ & \cdots & \circ \\
m+N+1 & m+N+2 & \cdots & m+N+r \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
m+(s-1)N+1 & m+(s-1)N+2 & \cdots & m+(s-1)N+r \\
\circ & \circ & \cdots & \circ
\end{array}$$

Observe that the S_n 's are all disjoint, since $N > r$. One can think of a flag as a rectangular array of s points $m+1, m+N+1, \dots, m+(s-1)N+1$ on a vertical pole, and with s horizontal strips $m+nN+1, m+nN+2, \dots, m+nN+r$, $n = 0, 1, \dots, (s-1)$.

Let $\Gamma = \{0, \lambda_1, \dots, \lambda_{d-1}\}$. It is easy to see that If $d > 1$, then a $d \times 1$ flag cannot be contained in $\mathbb{Z}(\widehat{\delta}_\Gamma) \cup \{0\}$. (by a simple Vandermonde argument). However, if S_0 is a strip of shorter length, we will consider several such strips in a flag configuration as above, and prove:

Theorem 4. *Fix two integers m, r with $[\frac{d}{2}] \leq r < d$. Suppose an $r \times d$ flag $F \subset \mathbb{Z}(\widehat{\delta}_\Gamma)$. Then Γ is rational.*

Proof. . Let $z_j = e^{2\pi i \lambda_j}$. The hypothesis implies that for each $k = 1, 2, \dots, r$, we have the following system of equations:

$$\begin{aligned}
\sum_{j=0}^{d-1} z_j^{m+k} &= 0 \\
\sum_{j=0}^{d-1} z_j^{m+k+N} &= 0 \\
&\vdots \\
\sum_{j=0}^{d-1} z_j^{m+k+(d-1)N} &= 0
\end{aligned}$$

Equivalently, for every $k = 1, 2, \dots, r$,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & z_1^N & \cdots & z_{d-1}^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_1^{(d-1)N} & \cdots & z_{d-1}^{(d-1)N} \end{pmatrix} \begin{pmatrix} 1 \\ z_1^{m+k} \\ \vdots \\ z_{d-1}^{m+k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus we can conclude that the above Vandermonde matrix is singular. Hence $z_i^N = z_j^N$ for some $i \neq j$. We define an equivalence relation $z_k \sim z_l \iff z_k^N = z_l^N$, and we write ρ_j as a representative of each equivalence class so obtained, $j = 0, 1, \dots, t$. Also let $[\rho_j] = \{z_{j1}, z_{j2}, \dots, z_{jl_j}\}$ be the set of elements (l_j in number) in the j th equivalence class. We can now extract a subsystem of the above system of equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \rho_1^N & \cdots & \rho_t^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_1^{(t-1)N} & \cdots & \rho_t^{(t-1)N} \end{pmatrix} \begin{pmatrix} \sum_1^{l_0} z_{0s}^{m+k} \\ \sum_1^{l_1} z_{1s}^{m+k} \\ \vdots \\ \sum_1^{l_{t-1}} z_{(t-1)s}^{m+k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now the ρ_j 's are all distinct, and hence the Vandermonde matrix on the left is non-singular. Hence, for each $k = 1, 2, \dots, r$ and $j = 0, 1, \dots, t$, we have

$$\sum_{s=1}^{l_j} z_{js}^{m+k} = 0.$$

Suppose $t > 1$, i.e. there are more than one equivalence classes $[\rho_j]$, then we can choose one equivalence class, say ρ_{j_0} which has less than or equal to $\lceil \frac{d}{2} \rceil + 1 \leq r$ elements. For this j_0 , we consider the first l_{j_0} equations from the above set of r equations. We have,

$$\begin{pmatrix} z_{j_0 1} & z_{j_0 2} & \cdots & z_{j_0 l_{j_0}} \\ z_{j_0 1}^2 & z_{j_0 2}^2 & \cdots & z_{j_0 l_{j_0}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_{j_0 1}^{l_{j_0}} & z_{j_0 2}^{l_{j_0}} & \cdots & z_{j_0 l_{j_0}}^{l_{j_0}} \end{pmatrix} \begin{pmatrix} z_{j_0 1}^m \\ z_{j_0 2}^m \\ \vdots \\ z_{j_0 l_{j_0}}^m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is a contradiction since the $z_{j_0 i}$'s are all distinct. Hence there is only one equivalence class. \square

Remark 2. If $\Lambda = \Gamma + \mathbb{Z}$ is a spectrum for a set $\Omega = A + [0, 1]$, with $|A| = d$, we know that $A - A \subset \widehat{\mathbb{Z}(\delta_\Gamma)}$. Since $|A - A| \leq \frac{d(d-1)}{2}$, an $r \times d$ flag with $\lfloor \frac{d}{2} \rfloor + 1 \leq r$ cannot be contained in $A - A$. In the next theorem we will improve Jager's result and show that rationality follows from the existence of a 'shorter' flag in the integer zero set $\widehat{\mathbb{Z}(\delta_\Gamma)}$. Hence if the set $A - A$ itself has enough structure to contain these shorter flags, then we can conclude the rationality of the spectrum.

We will now extend a result due to Tijdeman [18], which in our notation can be stated as

Theorem 5. [18] Let $r = d - 1$ and suppose that a $(d - 1) \times 2$ flag is contained in $\widehat{\mathbb{Z}(\delta_\Gamma)}$, then the extended $(d - 1) \times d$ flag is also contained in $\widehat{\mathbb{Z}(\delta_\Gamma)}$.

Our extension of this theorem is for smaller values of r .

Theorem 6. Suppose that an $r \times (d - r + 1)$ flag is contained in $\widehat{\mathbb{Z}(\delta_\Gamma)}$, then the extended $r \times d$ flag is also contained in $\widehat{\mathbb{Z}(\delta_\Gamma)}$.

Let $z_0 = 1, z_1, \dots, z_{d-1}$ be distinct complex numbers and

$$f_k = \sum_{j=0}^{d-1} z_j^k$$

Then the f_k s satisfy a d -term recurrence relation given by the Newton-Girard Formulae:

$$(1) \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{d-1} \end{pmatrix} \begin{pmatrix} f_k \\ f_{k+1} \\ \vdots \\ f_{d+k-1} \end{pmatrix} = \begin{pmatrix} f_{k+1} \\ f_{k+2} \\ \vdots \\ f_{k+d+1} \end{pmatrix}$$

In brief, we will write these equations as:

$$U\nu_k = \nu_{k+1}$$

Proof of Theorem 6. Let F be the $r \times (d-r+1)$ flag contained in $A-A \setminus \{0\} \subset \mathbb{Z}(\hat{\delta}_r)$. Then, each of the vectors $\nu_{m+1}, \nu_{m+N+1}, \dots, \nu_{m+(d-r)N+1}$ will be of the form given below:

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{pmatrix}$$

where the first r entries are 0. For ease of notation we will write $\mu_j = \nu_{m+jN+1}$, $j = 0, 1, \dots, (d-r)$. Clearly these $d-r+1$ vectors are linearly dependent, so there exist constants $\alpha_0, \alpha_1, \dots, \alpha_{d-r}$ such that

$$\mu_{d-r+1} = \alpha_0 \mu_0 + \alpha_1 \mu_1 + \dots + \alpha_{d-r} \mu_{d-r}$$

Now we apply the Newton-Girard matrix U N times to get

$$U^N \mu_j = \mu_{j+1}$$

Repeating this process, we see that all the vectors ν_{m+jN+1} are of the same form. But this means that the extended $r \times d$ flag is also contained in $\widehat{\mathbb{Z}(\delta_\Gamma)}$. \square

Combined with Theorem 4, we get

Theorem 7. *Let $\Lambda = \Gamma + \mathbb{Z}$ be a spectrum for a set $\Omega = A + [0, 1]$, with $A \subset \mathbb{Z}$ $|A| = d$. Suppose $r \geq \lfloor \frac{d}{2} \rfloor$, and an $r \times (d - r + 1)$ flag is contained in $A - A$, then Λ is rational.*

We have found that the existence of some variations of the flag patterns in $A - A$ again imply rationality of the spectrum.

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